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The stability of unbounded two- and three-dimensional flows subject to body forces: some exact solutions

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A formulation, previously employed to find exact Navier–Stokes solutions for planar disturbances in two- and three-dimensional flows with spatially uniform rates of strain, is here adapted to incorporate the contribution of various types of body force. In the absence of body forces, it is known that unbounded flows with constant vorticity and elliptical streamlines are unstable to certain planar disturbances, which are amplified by a Floquet mechanism. The influence of a Coriolis force upon this instability mechanism is here described in detail, as an illustration of the general formulation. The results are likely to be of geophysical interest and may also have relevance to the breakdown of closed-eddy structures in turbulence. The final section of the paper reviews other systems for which analogous exact solutions may be obtained.

1. Introduction

Craik & Criminale (1986) described a procedure for finding classes of exact solutions of the Navier–Stokes equations. These solutions consist of a 'basic flow' with spatially uniform rates of strain and a 'disturbance' of planar form: the disturbance is continuously distorted by the basic flow but nevertheless remains of planar form at all times. A somewhat similar formulation was given by Lagnado, Phan-Thien & Leal (1984); but this earlier work was restricted to two-dimensional basic flows and the authors were unaware that their linearized approximation is in fact an exact solution for single plane-wave modes. An early precursor of both papers is work of Kelvin (1887) on the linear stability of unbounded plane Couette flow.

Bayly (1986) was first to realize that the class of two-dimensional basic flows with closed elliptical streamlines can sustain a Floquet-type instability of certain plane-wave modes, owing to their periodic distortion. Bayly's results, though inviscid, admitted straightforward extension to incorporate viscosity, and this was subsequently done by Landman & Saffman (1987).

Though body-force terms were retained in the initial formulation of Craik & Criminale, none of the above studies derived results that incorporate the influence of body forces. However, there are some studies, restricted to disturbed unbounded plane Couette flow, that include a gravitational body force associated with density stratification (Phillips 1966; Hartman 1975; Knobloch 1984; Criminale & Cordova 1986), or Coriolis force (Yamagata 1976; Farrell 1982; Boyd 1983; Tung 1983; Criminale 1985; Shepherd 1985; Haynes 1987) or both (Criminale & Pinet-Plasencia 1985; Knobloch 1985). Furthermore, Craik (1988) has recently extended the Craik &

Criminale formulation to viscous magnetohydrodynamics (MHD) and has found classes of exact solutions in which MHD forces play a significant role.

The aims of this paper are twofold. The first is to give a general account of those flows influenced by body forces that admit exact solutions similar in kind to those enumerated by Craik & Criminale when body forces are absent. The second is to describe detailed results that extend Bayly's inviscid Floquet stability analysis of elliptical flows to incorporate a Coriolis force.

For this latter, the Coriolis effect is found to be marked. The results are of potential importance both to an understanding of geophysical phenomena and of closed-streamline structures within turbulent flows. It is found that, for all but a narrow band of rotation speeds, elliptical-vortex flows are inviscidly unstable to three-dimensional plane-wave disturbances. However, for some rotation speeds, the instability is much stronger than for others and the weaker instability may often be suppressed by viscous dissipation.

2. General formulation

To avoid undue repetition, the notation of Craik & Criminale (1986, henceforth referred to as I) is retained and only the key results are given here. The present analysis closely follows that of the earlier work, but includes additional body-force terms. The governing equations are the incompressible Navier–Stokes equations,

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\mathbb{F}_i+\nu\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}, \tag{2.1a}$$

$$\frac{\partial u_j}{\partial x_j}=0, \tag{2.1b}$$

here expressed in Cartesian tensor form with summation over repeated lower indices. The flow field is u_i $(i=1,2,3)$, the pressure p, the density ρ and kinematic viscosity ν. $\mathbb{F}_i$ denotes an as yet unspecified body force per unit mass. Cartesian space coordinates are $x_i\,(i=1,2,3)$ and t denotes time.

Solutions are sought in which the velocity field may be decomposed into a 'basic flow' $\mathscr{U}_i$ and a 'disturbance' u_i', as $u_i=\mathscr{U}_i+u_i'$; but no assumptions regarding the relative sizes of $\mathscr{U}_i$ and u_i' are implied. Since the basic flow itself satisfies the Navier–Stokes equations,

$$\frac{\partial \mathscr{U}_i}{\partial t}+\mathscr{U}_j\frac{\partial \mathscr{U}_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial x_i}+\mathbb{F}_i^{(0)}+\nu\frac{\partial^2 \mathscr{U}_i}{\partial x_j\,\partial x_j}, \tag{2.2a}$$

$$\frac{\partial \mathscr{U}_j}{\partial x_j}=0, \tag{2.2b}$$

where $p^{(0)}$ and $\mathbb{F}_i^{(0)}$ respectively denote the 'basic' pressure and body force.

It is necessary to restrict attention to basic flows for which $\mathscr{U}_i$ depends linearly on the space coordinates, as

$$\mathscr{U}_i=\sigma_{ij}(t)\,x_j+\mathscr{U}_i^{(0)}(t)\quad(i,j=1,2,3). \tag{2.3}$$

Similarly, the basic body force and pressure are here taken in the form

$$\mathbb{F}_i^{(0)}=f_{ij}^{(0)}\,x_j+\mathscr{F}_i^{(0)}(t), \tag{2.4a}$$

$$p^{(0)}=p_0^{(0)}+\pi_j^{(0)}\,x_j+\pi_{ij}^{(0)}\,x_i\,x_j. \tag{2.4b}$$

It follows that the various quantities are related by

$$\frac{\mathrm{d}\mathscr{U}_i^{(0)}}{\mathrm{d}t}+\sigma_{ij}\mathscr{U}_j^{(0)}=\mathscr{F}_i^{(0)}-\rho^{-1}\pi_i^{(0)}, \tag{2.5a}$$

$$\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}t}+\sigma_{ik}\sigma_{kj}-f_{ij}^{(0)}=-\rho^{-1}(\pi_{ij}^{(0)}+\pi_{ji}^{(0)}), \tag{2.5b}$$

$$\sigma_{jj}=0. \tag{2.5c}$$

The second of these relations may be re-expressed in matrix form as

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}+\boldsymbol{S}^2-\boldsymbol{F}=\boldsymbol{M}(t); \qquad \boldsymbol{S}\equiv\{\sigma_{ij}\}, \quad \boldsymbol{F}\equiv\{f_{ij}^{(0)}\}, \tag{2.6}$$

where $\boldsymbol{S}$ has zero trace by virtue of (2.5*c*) and $\boldsymbol{M}(t)$ may be regarded as an arbitrary symmetric matrix. This differs from equation (2.4) of I only by the addition of the term in $\boldsymbol{F}$.

It may be noticed at once that, if $\mathbb{F}_i^{(0)}$ is conservative, $\boldsymbol{F}$ must be symmetric and so may be absorbed into the arbitrary $\boldsymbol{M}(t)$. It follows that, with symmetric $\boldsymbol{F}$, the same class of admissible velocity fields arises as in I. In particular, those independent of time have matrix $\boldsymbol{S}$ reducible to one of the forms

$$\{\sigma_{ij}\}=\boldsymbol{S}=\begin{pmatrix}0 & 0 & 0\\ 0 & \epsilon & h\\ 0 & -h & -\epsilon\end{pmatrix} \text{ or } \begin{pmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & -a-b\end{pmatrix} \tag{2.7a, b}$$

when described relative to suitably chosen principal axes: cf. I §3.

Time-dependent basic states are discussed later, in §4: unsteadiness is the inevitable consequence whenever vortex lines are stretched by the irrotational part of the flow. In fact, any one of this general class of divergence-free basic flows with uniform rates of strain admits solution of the evolution equations for planar disturbances; but the steady basic flows hold most physical interest. Though artificial in the sense that they are unbounded in space and attain unbounded velocity at infinity, they may be regarded as valid local approximations to flows near stagnation points. Streamlines are open for (2.7*b*) and for (2.7*a*) with $h^2<\epsilon^2$; closed for (2.7*a*) with $h^2>\epsilon^2$.

We now focus attention on a class of flows for which $\boldsymbol{F}$ is not symmetric: namely, those subject to a Coriolis force when viewed in a reference frame rotating with constant angular velocity $-\boldsymbol{\Omega}=-(\Omega_1,\Omega_2,\Omega_3)$ relative to some inertial frame. For these.

$$\mathbb{F}_i^{(0)}=2(\boldsymbol{\Omega}\times\mathscr{U})_i=2\epsilon_{ijk}\Omega_j(\sigma_{kl}x_l+\mathscr{U}_k^{(0)}), \tag{2.8}$$

where ϵ_{ijk} is the permutation tensor. Accordingly,

$$\boldsymbol{F}=\{f_{il}^{(0)}\}=\{2\epsilon_{ijk}\Omega_j\sigma_{kl}\}, \quad \mathscr{F}_i^{(0)}=2\epsilon_{ijk}\Omega_j\mathscr{U}_k^{(0)} \tag{2.9}$$

and (2.5*a*, *b*) give

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}+(\boldsymbol{S}-\boldsymbol{G})\boldsymbol{S}=\boldsymbol{M}, \tag{2.10a}$$

$$\frac{\mathrm{d}\mathscr{U}^{(0)}}{\mathrm{d}t}+(\boldsymbol{S}-\boldsymbol{G})\mathscr{U}^{(0)}=-\rho^{-1}\boldsymbol{\pi}^{(0)} \tag{2.10b}$$

with $\mathscr{U}^{(0)} \equiv \mathscr{U}_i^{(0)}$, $\boldsymbol{\pi}^{(0)} \equiv \pi_i^{(0)}$ and

$$\boldsymbol{G} \equiv 2\begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

For steady basic flows, $(\boldsymbol{S}-\boldsymbol{G})\boldsymbol{S}$ must be symmetric. This requirement reduces, after some algebra, to

$$\boldsymbol{S}\begin{pmatrix} \Omega_1+c \\ \Omega_2+d \\ \Omega_3+e \end{pmatrix} = 0, \tag{2.11}$$

with $\boldsymbol{S}$ taken in the general form

$$\boldsymbol{S} = \begin{pmatrix} a & e & -d \\ -e & b & c \\ d & -c & -(a+b) \end{pmatrix} \tag{2.12}$$

appropriate for axes chosen along the principal directions of rate-of-strain.

For (2.11) to have a solution, it is necessary that either the determinant of $\boldsymbol{S}$ should vanish or that

$$(\Omega_1, \Omega_2, \Omega_3) = -(c, d, e). \tag{2.13}$$

In the latter case, the net basic vorticity in the non-rotating inertial frame is identically zero and the basic velocity field in this frame is a pure rate of strain at each instant. However, if the strain rates a, b and $-(a+b)$ are constants in the rotating frame (as will be assumed subsequently) the principal axes of rate-of-strain rotate with angular velocity $-\boldsymbol{\Omega}$ relative to the inertial frame.

If, on the other hand, the determination of $\boldsymbol{S}$ is zero, then the strain rate a may be found in terms of b, c, d, e as the solutions of the quadratic equation

$$a^2b + a(e^2+b^2-c^2) + b(e^2-d^2) = 0. \tag{2.14}$$

Since the roots a must be real, it is necessary that

$$(e^2+b^2-c^2)^2 > 4b^2(e^2-d^2)$$

if det $\boldsymbol{S}$ is to be zero. In turn, (2.11) yields two independent linear equations connecting the three Ω_i: accordingly, $|\Omega|$ may be arbitrarily prescribed and the three associated components determined by these relations.

If one of the principal strain rates, say a, is taken to be zero, then $\boldsymbol{S}$ has the form

$$\boldsymbol{S} = \begin{pmatrix} 0 & e & -d \\ -e & \epsilon & c \\ d & -c & -\epsilon \end{pmatrix}$$

(where b is now replaced by ϵ) and its determinant vanishes provided

$$\epsilon(d^2-e^2) = 0. \tag{2.15}$$

If $\epsilon = 0$, the motion is a pure rotation and (2.11) yields

$$(\Omega_1, \Omega_2, \Omega_3) = K(c, d, e), \tag{2.16}$$

where K is an arbitrary constant. Alternatively, if (2.15) is satisfied because $d = \pm e$, then

$$\frac{\Omega_1 \mp \epsilon}{c \pm \epsilon} = \frac{\pm \Omega_2}{e} = \frac{\Omega_3}{e} = K \tag{2.17}$$

for arbitrary K. Corresponding relations for three non-zero strain rates are omitted for brevity.

For the above classes of admissible basic flows, the evolution of planar disturbances is exactly governed by linear equations, much as in I. The disturbance velocity u_i' and pressure $p^{(1)}$ have the form

$$\begin{pmatrix} u_i' \\ p^{(1)} \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} \hat{u}_i(t) \\ \hat{p}(t) \end{pmatrix} \exp\left[\mathrm{i}\alpha_j(t)x_j + \mathrm{i}\delta(t)\right] \right\} \tag{2.18}$$

with time-dependent wavenumber $\boldsymbol{\alpha}(t) \equiv \{\alpha_i(t)\}$. The formulation differs from I only by inclusion of Coriolis terms. In particular, the disturbance Coriolis force is

$$2(\boldsymbol{\Omega} \times \boldsymbol{u}') = \mathrm{Re}\,\{2\epsilon_{ijk}\Omega_j \hat{u}_k \exp\left[\mathrm{i}\alpha_l x_l + \mathrm{i}\delta\right]\}. \tag{2.19}$$

Just as in I, it is necessary to choose

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} + \alpha_j \mathscr{U}_j^{(0)} = 0, \tag{2.20a}$$

$$\frac{\mathrm{d}\alpha_j}{\mathrm{d}t} + \alpha_k \sigma_{kj} = 0, \quad \text{i.e.} \quad \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{\alpha} = 0, \tag{2.20b}$$

where $\boldsymbol{S}^{\mathrm{T}}$ is the transpose of $\boldsymbol{S}$. This leaves the momentum and continuity equations in the form

$$\frac{\mathrm{d}\hat{\boldsymbol{u}}}{\mathrm{d}t} + \boldsymbol{S}\hat{\boldsymbol{u}} + \nu(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})\,\hat{\boldsymbol{u}} = -\frac{\mathrm{i}\boldsymbol{\alpha}\hat{p}}{\rho} + 2(\boldsymbol{\Omega}\times\hat{\boldsymbol{u}}), \tag{2.21a}$$

$$\hat{\boldsymbol{u}}\cdot\boldsymbol{\alpha} = 0. \tag{2.21b}$$

Elimination of $\hat{p}$ yields

$$\hat{p}/\rho = \frac{\mathrm{i}\boldsymbol{\alpha}^{\mathrm{T}}(2\boldsymbol{S}-\boldsymbol{G})\hat{\boldsymbol{u}}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}, \tag{2.22a}$$

$$\frac{\mathrm{d}\hat{u}_i}{\mathrm{d}t} + \sigma_{ij}\hat{u}_j + \nu(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})\,\hat{u}_i = \frac{\alpha_i\alpha_j\hat{u}_k(2\sigma_{jk} + 2\Omega_l\epsilon_{jkl})}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} + 2\epsilon_{ijk}\Omega_j\hat{u}_k, \tag{2.22b}$$

the latter of which is just

$$\frac{\mathrm{d}\hat{\boldsymbol{u}}}{\mathrm{d}t} + \boldsymbol{T}\hat{\boldsymbol{u}} = 0, \quad \boldsymbol{T} \equiv \{\tau_{ij}\}, \tag{2.23}$$

$$\tau_{ij} = \nu(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})\,\delta_{ij} + \sigma_{ij} - \frac{2\alpha_i\alpha_k\sigma_{kj}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} + \frac{2\epsilon_{ljk}\Omega_k}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}[(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})\,\delta_{il} - \alpha_i\alpha_l].$$

Here $\boldsymbol{T}$ is a time-dependent matrix which is a known function of the elements σ_{ij} of a permissible $\boldsymbol{S}$ and of the time-varying wavenumber $\boldsymbol{\alpha}$ governed by (2.20b). The wavenumber equation is identical to that of I and the structure of (2.23) is similar but contains additional terms in the Ω_i. Analytical solutions may be found for special cases but computation is more often necessary from this point.

3. Two-dimensional elliptical basic flows

We here restrict attention to the family of admissible two-dimensional basic flows with matrices $\boldsymbol{S}$ of the form

$$\boldsymbol{S} = \{\sigma_{ij}\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & c \\ 0 & -c & -\epsilon \end{pmatrix}, \quad (d = e = 0) \tag{3.1}$$

as seen relative to a frame rotating with angular velocity

$$-\boldsymbol{\Omega} = -(\Omega_1, 0, 0), \quad (\Omega_2 = \Omega_3 = 0). \tag{3.2}$$

The corresponding class of flows with $\boldsymbol{\Omega} = 0$ has been examined in earlier papers. When $c^2 < \epsilon^2$, the streamlines are hyperbolic and the wavenumber varies exponentially with time: see Lagnado *et al.* (1984) and I. The continuous stretching and tilting by the basic flow then often causes the disturbance to amplify, but the consequent shortening of the disturbance wavelength ensures that viscous dissipation ultimately predominates (except for isolated special cases in which the wavelength continuously increases with time). When $c^2 = \epsilon^2$, the basic flow is a parallel plane Couette flow and the wavenumber depends algebraically on t: this was examined by Kelvin (1887). When $c^2 > \epsilon^2$, the basic flow has elliptical streamlines and the disturbance wavenumber varies periodically with t. This was examined in I and more comprehensively by Bayly (1986) and Landman & Saffman (1987). Bayly was first to realize that, though the wavenumber is periodic, the disturbance amplitude is not necessarily so. For a range of wavenumbers, the disturbance undergoes a Floquet-type instability due to the periodic forcing. Bayly's inviscid analysis of this instability was extended by Landman & Saffman to include the damping influence of viscosity.

As also described in equation (3.14) of I, the viscous terms may be removed by the transformation

$$\check{u}_i(t) = \exp\left\{-\nu \int_0^t (\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})\,\mathrm{d}t\right\}\check{u}_i(t) \tag{3.3}$$

and the resultant variables $\check{u}_i(t)$ satisfy the corresponding inviscid equations

$$\frac{\mathrm{d}\check{\boldsymbol{u}}}{\mathrm{d}t} + \boldsymbol{T}_{\mathrm{in}}\check{\boldsymbol{u}} = 0, \quad \boldsymbol{T}_{\mathrm{in}} = \{\hat{\tau}_{ij}\}$$

where the elements $\hat{\tau}_{ij}$ are as given for τ_{ij} in (2.23) but with viscosity ν set equal to zero. Just as in I, the first column of $\boldsymbol{T}_{\mathrm{in}}$ is identically zero: it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\check{u}_2\\ \check{u}_3\end{pmatrix} = \begin{pmatrix}-\epsilon + r\alpha_2 & -c - 2\Omega_1 + s\alpha_2\\ c + 2\Omega_1 + r\alpha_3 & \epsilon + s\alpha_3\end{pmatrix}\begin{pmatrix}\check{u}_2\\ \check{u}_3\end{pmatrix} \equiv \boldsymbol{Q}(t)\begin{pmatrix}\check{u}_2\\ \check{u}_3\end{pmatrix},$$

$$r \equiv \frac{2[\alpha_2\epsilon - \alpha_3(c+\Omega_1)]}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}, \quad s \equiv \frac{2[\alpha_2(c+\Omega_1) - \alpha_3\epsilon]}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} \tag{3.4}$$

and $\check{u}_1$ is most easily recovered from the continuity equation (2.21*b*),

$$\alpha_1\check{u}_1 + \alpha_2\check{u}_2 + \alpha_3\check{u}_3 = 0. \tag{3.5}$$

The wavenumber $\boldsymbol{\alpha}(t)$ satisfying (2.20*b*) with $\boldsymbol{S}$ as in (3.1) is given in I; namely

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & A & -C\\ 0 & C & B\end{pmatrix}\begin{pmatrix}\alpha_{10}\\ \alpha_{20}\\ \alpha_{30}\end{pmatrix}, \quad \left.\begin{array}{l} A \equiv \cos\beta t - (\epsilon/\beta)\sin\beta t,\\ B \equiv \cos\beta t + (\epsilon/\beta)\sin\beta t,\\ C \equiv -(c/\beta)\sin\beta t,\end{array}\right\} \tag{3.6}$$

$$\beta \equiv (c^2 - \epsilon^2)^{\frac{1}{2}}, \quad (c^2 > \epsilon^2),$$

where $\boldsymbol{\alpha}_0 = (\alpha_{10}, \alpha_{20}, \alpha_{30})$ denotes the wavenumber at $t = 0$. The time-dependent matrix $\boldsymbol{Q}(t)$ in (3.4) is therefore known explicitly.

The inviscid stability problem is governed by (3.4) and may be solved by Floquet theory as described by Bayly (1986). The present work extends Bayly's problem to incorporate rotation of the coordinate frame in which the elliptical flow appears as steady. The solution is some linear combination of two Floquet modes of the form

$$(\check{u}_2, \check{u}_3) = \mathrm{e}^{\sigma t}(v_2, v_3), \quad v_j(t+2\pi\beta^{-1}) = v_j(t) \quad (j = 1, 2), \tag{3.7}$$

where the v_j have period $\tau \equiv 2\pi/\beta$ that matches that of the wavenumber. The σ denote two Floquet exponents as yet unknown. The associated Floquet matrix problem is

$$\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}t} = \boldsymbol{Q}(t)\boldsymbol{N}, \quad \boldsymbol{N}(0) = \boldsymbol{I} \tag{3.8}$$

and this must be solved numerically to find the 2×2 matrix $\boldsymbol{N}$ at time $\tau = 2\pi/\beta$. This yields its two eigenvalues, the Floquet multipliers μ_j, which in turn give the two Floquet exponents as

$$\sigma = \sigma_{1,2} = (\beta/2\pi)\log\mu_j \quad (j = 1, 2). \tag{3.9}$$

It may be shown that the determinant of $\boldsymbol{N}(\tau)$ must equal unity. This property provides a ready check on the accuracy of the numerical procedure used to calculate $\boldsymbol{N}(\tau)$ and hence the μ_j. Because the determinant of $\boldsymbol{N}(\tau)$ is unity, the product $\mu_1\mu_2$ of its two eigenvalues is also unity. Since $\boldsymbol{N}$ is a real matrix, this means that (μ_1, μ_2) are either real and positive, of the form $(q, 1/q)$ or are complex conjugates of form $\exp(\pm\mathrm{i}\phi)$ with unit modulus (see Bayly 1986).

As the two eigenvalues of $\boldsymbol{N}(\tau)$ are the roots μ_j of

$$\mu^2 - \mu(\operatorname{tr}\boldsymbol{N}) + \det\boldsymbol{N} = 0$$

and as $\det\boldsymbol{N}(\tau) = 1$, it follows that

$$\mu_{1,2} = \tfrac{1}{2}\operatorname{tr}\boldsymbol{N} \pm [\tfrac{1}{4}(\operatorname{tr}\boldsymbol{N})^2 - 1]^{\frac{1}{2}}, \tag{3.10}$$

where the trace of $\boldsymbol{N}$ is evaluated at $t = \tau$. Instability requires $\sigma > 0$ and this occurs, for the greater real root μ_j, whenever the trace of $\boldsymbol{N}(\tau)$ is greater than 2.

Since α_3 passes through zero once during each period τ, there is no loss of generality in choosing the initial instant $t = 0$ as that for which $\alpha_3 = 0$ and $\alpha_2 \geqslant 0$. (However, note that this initialization differs from the one chosen by Bayly.) This allows simplification of (3.6) to

$$\alpha_1(t) = \alpha_{10}, \quad \alpha_2(t) = (\cos\beta t - \epsilon\beta^{-1}\sin\beta t)\,\alpha_{20}, \quad \alpha_3(t) = (-c\beta^{-1}\sin\beta t)\,\alpha_{20}. \tag{3.11}$$

Also, since the inviscid problem has no natural length or time scales, these may be chosen arbitrarily: convenient choices are those that give

$$c = 1, \quad |\boldsymbol{\alpha}(0)| = 1.$$

Accordingly, we set

$$\left.\begin{aligned} &c = 1, \quad \beta = (1-\epsilon^2)^{\frac{1}{2}} \quad (0 \leqslant \epsilon < 1), \\ &\alpha_{30} = 0, \quad \alpha_{20} = (1-\alpha_{10}^2)^{\frac{1}{2}} \quad (0 \leqslant \alpha_{10} \leqslant 1) \end{aligned}\right\} \tag{3.12}$$

without loss. The basic flows are then parametrized by ϵ. This is a measure of the eccentricity of the elliptical streamlines and is related to Bayly's parameter E by

$$E = \left(\frac{1\pm\epsilon}{1\mp\epsilon}\right)^{\frac{1}{2}},$$

where E is the ratio of the principal axes of the ellipses. Thus, $\epsilon = 0$ is a pure rotation and the eccentricity increases indefinitely as ϵ approaches 1; for instance, the ratio of major to minor axes of streamlines is 3 when $\epsilon = 0.8$. The background rotation is parametrized by Ω_1 (now with timescale such that $c = 1$) and the initial disturbance wavenumber by α_{10}. With these modifications, the Floquet matrix $\boldsymbol{Q}$ is somewhat simplified.

Numerical integration of (3.8), to find $\boldsymbol{N}(2\pi/\beta)$, was carried out on a BBC microcomputer using a general-purpose differential equations package developed by Mr I. Ellery. Accuracy could easily be assessed by evaluating the determinant of $\boldsymbol{N}(2\pi/\beta)$: in almost every case examined, this determinant was very close to the known correct value of unity and was within the stated accuracy of the package. In just a few limiting cases of no particular importance, the matrix $\boldsymbol{N}(2\pi/\beta)$ was ill-conditioned and errors became significant. All such cases arose when α_{10} was close to (but less than) unity and the eccentricity was large: fortunately, the case $\alpha_{10} = 1$ has an analytic solution and these doubtful numerical results could be discarded. Accordingly, the overall accuracy of the results obtained was considered satisfactory.

The case $\alpha_1(0) = 1$, $\alpha_2(0) = 0$ simply has $\boldsymbol{\alpha} = (1, 0, 0)$ at all times. Then, $r = s = 0$ and $\boldsymbol{Q}(t)$ reduces to the constant matrix

$$\begin{pmatrix} -\epsilon & -1-2\Omega \\ 1+2\Omega_1 & \epsilon \end{pmatrix}$$

(with c taken as unity). The solutions for $\boldsymbol{N}(t)$ are then exponential, in $\exp(\sigma t)$, where σ corresponds to the Floquet exponents defined in (3.9). It is readily found that, for this case,

$$\sigma_{1,2} = \pm[\epsilon^2 - (1+2\Omega_1)^2]^{\frac{1}{2}}. \tag{3.13}$$

Clearly, one mode is unstable whenever $-\epsilon < 1+2\Omega_1 < \epsilon$. For example, with $\epsilon = 0.8$, such modes are unstable whenever $-0.9 < \Omega_1 < -0.1$; but small values of ϵ confine the instability to a narrow band centred on $\Omega_1 = -0.5$. This simple result provided a further useful check on the computations.

A sample of the numerical results is shown in figures 1(*a–f*). These show the unstable Floquet exponent ($\sigma > 0$) for various constant strain rates ϵ and background angular velocities Ω_1. The horizontal axis shows the normalized initial wavenumber component α_{10} and the vertical axis shows σ. It is clear that increasing the eccentricity of the streamlines increases the band of unstable wavenumbers α_{10}, and that the rotation Ω_1 has a strong effect. Results for $\Omega_1 = 0$ are in good agreement with those of Bayly (1986), when account is taken of the different initializations. Notice, in particular, that there is a range of values of Ω_1, for which the most unstable mode has $\alpha_{10} = 1$: that is, the case solved exactly above. It is also noteworthy that the instability at negative values of Ω_1 is typically much weaker than for positive Ω_1; that it is confined to a narrower range of wavenumbers; and that it is absent altogether at some values of Ω_1.

By tediously determining many such curves for various fixed ϵ and Ω_1, the values of α_{10} at which σ first becomes real and positive may be mapped out. This gives a stability boundary in three-dimensional (α_{10}, ϵ, Ω_1)-space. The section of this surface for $\epsilon = 0.8$ is shown in figure 2.

At smaller values of ϵ the larger of the unstable regions is narrower. For instance, that for $\epsilon = 0.6$ (not shown) lies within the corresponding region shown for $\epsilon = 0.8$: its unstable band of rotation speeds is confined to $-1.6 < \Omega_1 < -0.4$ in the limiting case $\alpha_{10} = 1$. Correspondingly, when $\epsilon = 0.6$, the smaller unstable region extends somewhat less far to the right, includes a narrower band of wavenumbers and has lower maximum growth rates at each fixed Ω_1, than for $\epsilon = 0.8$. Unexpectedly, the smaller unstable region for $\epsilon = 0.6$ does not lie totally within that for $\epsilon = 0.8$, but lies slightly above it. For example, at $\Omega_1 = -2$, the unstable band of wavenumbers is approximately $0.425 < \alpha_{10} < 0.465$ for $\epsilon = 0.6$ while that for $\epsilon = 0.8$ is about $0.34 < \alpha_{10} < 0.425$; similarly, at $\Omega_1 = -1.7$, the corresponding unstable bands are close to (0.58, 0.63) and (0.485, 0.565) respectively.

It is particularly noteworthy that there is instability, for some wavenumbers at each ϵ, for all but a rather narrow range of rotation speeds Ω_1. For $\epsilon = 0.8$ the stable range is $-1.3 < \Omega_1 < -0.9$ approximately and this range widens as ϵ is reduced, tending to $-1.5 < \Omega_1 < -0.5$ as ϵ approaches zero. Naturally, the maximum growth rate of unstable modes also shrinks to zero as the state of pure rotation is approached. At the other extreme, the eccentricity increases without limit as ϵ approaches 1 from below and the true limiting case is unbounded plane Couette flow in a rotating frame. The stability of this flow has been studied by Farrell (1982), Criminale (1985) and others.

Figure 3 shows some typical results for the temporal evolution of disturbances. Cases (*a*) and (*b*) exhibit doubly periodic behaviour of stable inviscid modes with pure imaginary Floquet exponents $\sigma_{1,2}$: these would eventually be damped by viscosity. Case (*c*) shows inviscid instability with real $\sigma_1 > 0$.

A physical understanding of the instability mechanism may be gained by first considering the case of small eccentricity. The streamlines of the basic flow are then virtually circular and this near solid-body rotation can support inertial waves with frequencies

$$\omega = \pm 2(\boldsymbol{\Omega}' \cdot \boldsymbol{\alpha})$$

relative to the rotating fluid. Here, $\boldsymbol{\Omega}'$ includes both the background rotation $-\boldsymbol{\Omega}$ and the contribution of the constant vorticity $(-2c, 0, 0)$ with $c = 1$: accordingly, $\boldsymbol{\Omega}' = -(\Omega_1 + 1, 0, 0)$. As originally pointed out by Bayly (1986) for the case $\Omega_1 = 0$, these inertial waves are in fact subject to periodic extension and contraction by the shearing component of the basic flow. For most wavenumbers this simply produces an additional periodic modulation; but others can be driven unstable when the forcing frequency (as observed by fluid particles) is close to that of the inertial-wave frequency.

Now, the former forcing frequency is the same as that of the periodically varying wavenumber vector: i.e. $c = 1$ for $\epsilon = 0$ or β for $\epsilon \neq 0$. Accordingly, instability can be expected to occur close to those wavenumbers with

$$\pm 2(\Omega_1 + 1)\alpha_{10} = (1 - \epsilon^2)^{\frac{1}{2}} \equiv \beta; \tag{3.14}$$

these lie on both branches of a hyperbola, for each fixed ϵ. As figure 2 shows, this is indeed so. To aid comparison, points labelled with plus signs and circles are inserted in the figure. Those marked by a plus sign lie on the hyperbola $(\Omega_1 + 1)\alpha_{10} = \pm 0.5$ corresponding to $\epsilon = 0$ and denote pure (neutrally stable) inertial waves; while those marked by a circle are on the hyperbola $(\Omega_1 + 1)\alpha_{10} = \pm 0.3$ corresponding to elliptical flow with $\epsilon = 0.8$. The downwards shifting of the regions of instability, more marked for the narrower region, as ϵ increases is obviously associated with the decrease in frequency β as ϵ increases. A more detailed physical picture of the mechanism can be constructed in terms of the periodic extension and contraction of vortex lines associated with the propagating plane-wave disturbance.

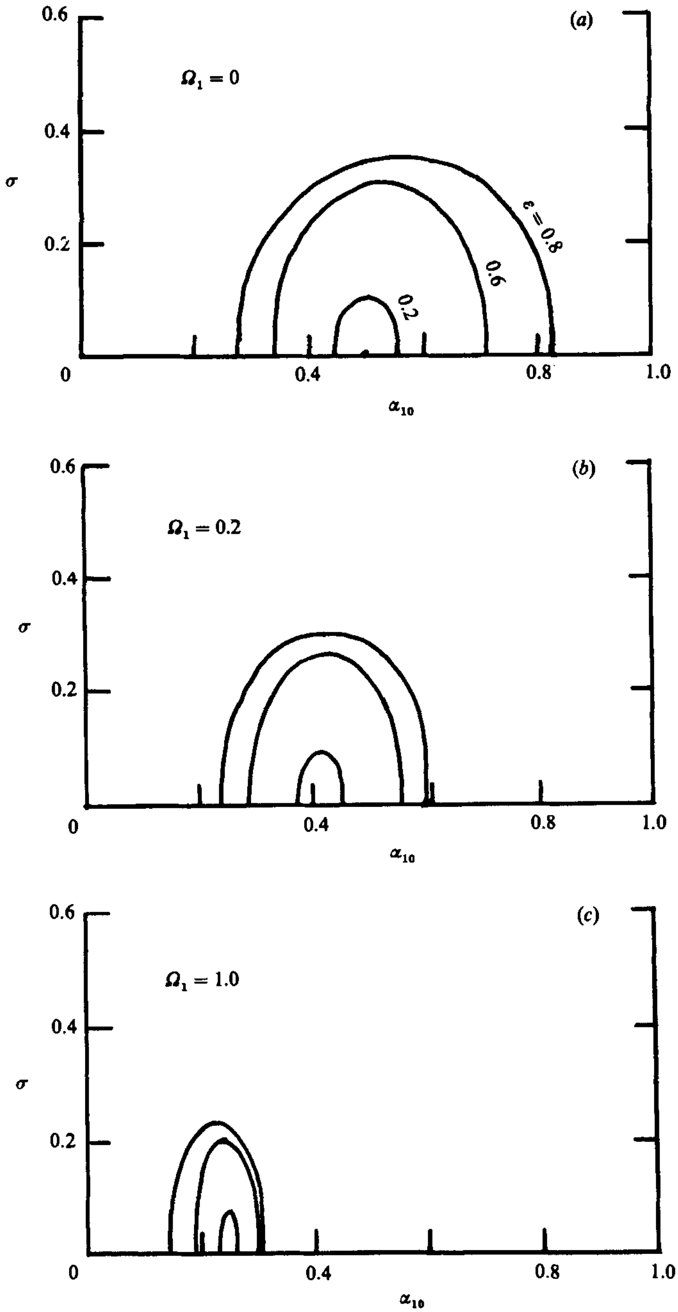

FIGURE 1 (a–c). For caption see facing page.

The modification by viscosity of these inviscid results is effected by the transformation (3.3). Substitution from (3.11) and integration over the rotation period $\tau = 2\pi/\beta$ readily yields the mean viscous decay rate to be

$$\sigma_{\text{visc}} = -\frac{\nu(1-\epsilon^2\alpha_{10}^2)}{1-\epsilon^2}, \tag{3.15}$$

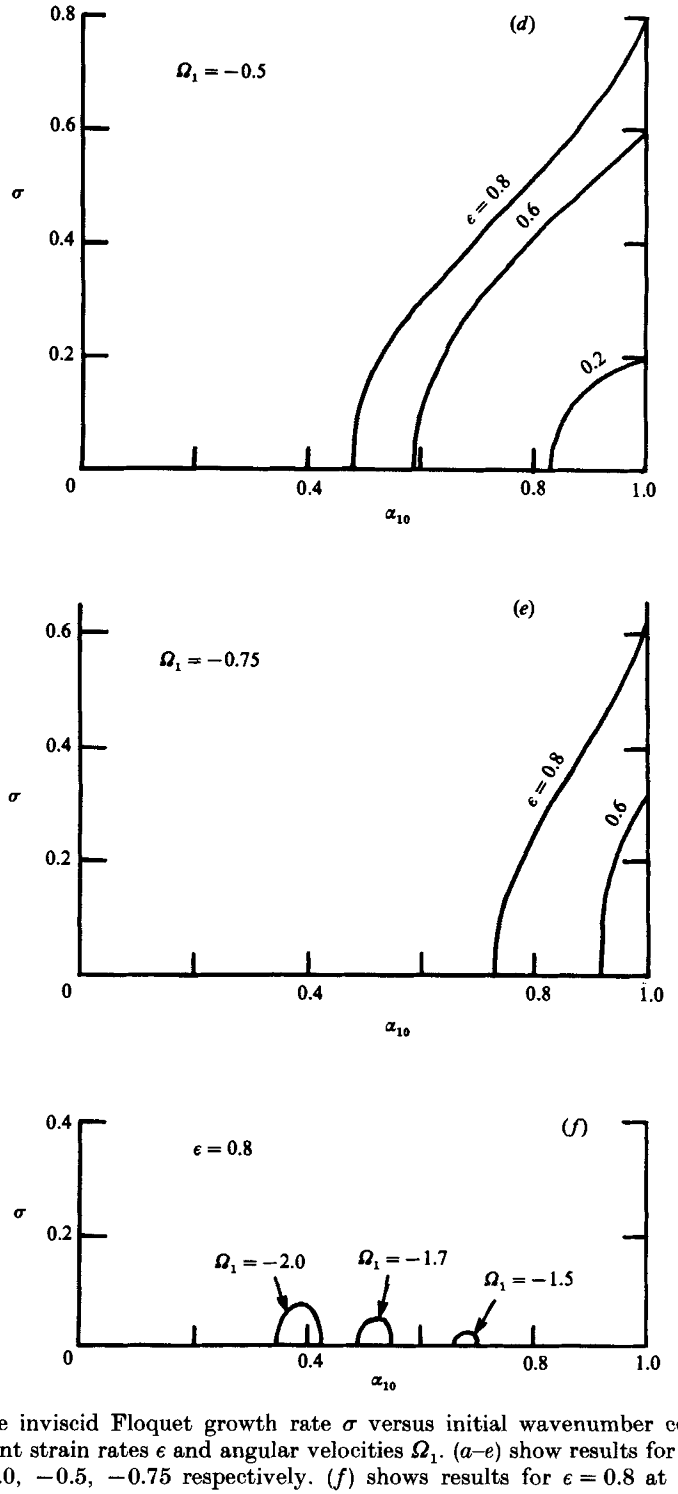

FIGURE 1. The inviscid Floquet growth rate σ versus initial wavenumber component α_{10}, for various constant strain rates ϵ and angular velocities Ω_1. (*a–e*) show results for $\epsilon = 0.2, 0.6, 0.8$ at $\Omega_1 = 0, 0.2, 1.0, -0.5, -0.75$ respectively. (*f*) shows results for $\epsilon = 0.8$ at $\Omega_1 = -1.5, -1.7, -2.0$.

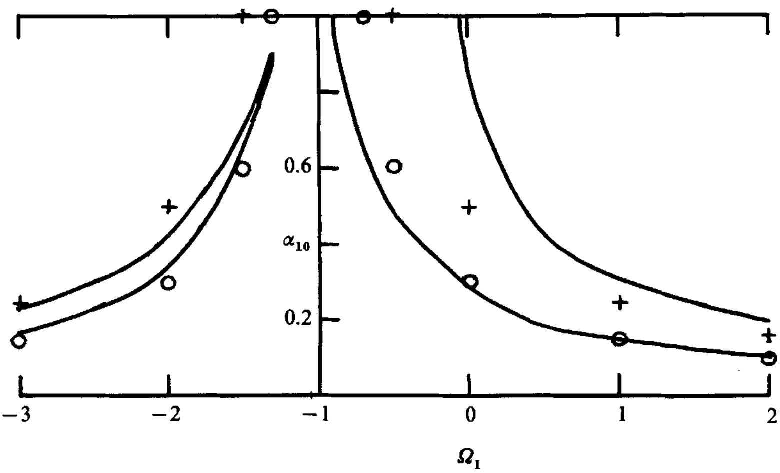

FIGURE 2. Inviscid instability boundaries for $\epsilon = 0.8$ in the (Ω_1, α_{10})-plane. Two distinct unstable bands exist: that on the right exhibits largest growth rates. A narrow range of Ω_1 near -1 is completely stable. Points marked + lie on the hyperbola $(\Omega_1+1)\alpha_{10} = \pm 0.5$ and those marked ○ lie on $(\Omega_1+1)\alpha_{10} = \pm 0.3$ (see text).

which must be added to the Floquet exponent σ in (3.7) to find the corresponding viscous result. However, because of the normalizations of c and of the lengthscale of the disturbances, which are equivalent to defining new units of length and time, the so-scaled value of ν used in (3.15) is actually $\nu|\alpha(0)|^2/c$ in unscaled units. Though this result appears, at first sight, to differ from that of Landman & Saffman (1987) for the case $\Omega_1 = 0$, this difference is solely attributable to the different normalizations and initializations used.

It is clear that short-wave disturbances are always strongly damped by viscous action but that the inviscid instability must persist for sufficiently long waves. As the inviscid growth rates of the narrower instability band are typically much less that for the wider one, viscous damping will suppress the narrower band at wavelengths that are still unstable in the wider. This result is of particular relevance when one considers possible extension of the theory to deal with elliptic vortices of finite extent, or to those with constant vorticity only in a finite central core. Such extension will be far from straightforward, and it is perhaps premature (*pace* Landman & Saffman's §4) to attempt comparison between the present theory and experiments that show evidence of three-dimensional distortion and breakup of finite vortex structures.

Nevertheless, some corroboration is provided by the numerical computations of Pierrehumbert (1986) for inviscid disturbed elliptical flows with $\Omega_1 = 0$; also, in the remoter context of unstable shear flow, by computational results of Metcalfe *et al.* (1987). The present growth industry of direct numerical simulation of complex flows is providing much data that requires physical interpretation. For instance, representations of two-dimensional 'turbulence' (McWilliams 1984; Babiano *et al.* 1987) show the development and persistence of strong coherent circular vortices; but elliptical vortices rarely last for long. However, it is risky to ascribe diverse phenomena to a common cause. The stability or instability of wall-influenced flows, such as the thin 'cats' eyes' of finite Tollmien–Schlichting waves, is probably better explained in terms of weakly nonlinear mode interactions than by vague analogy

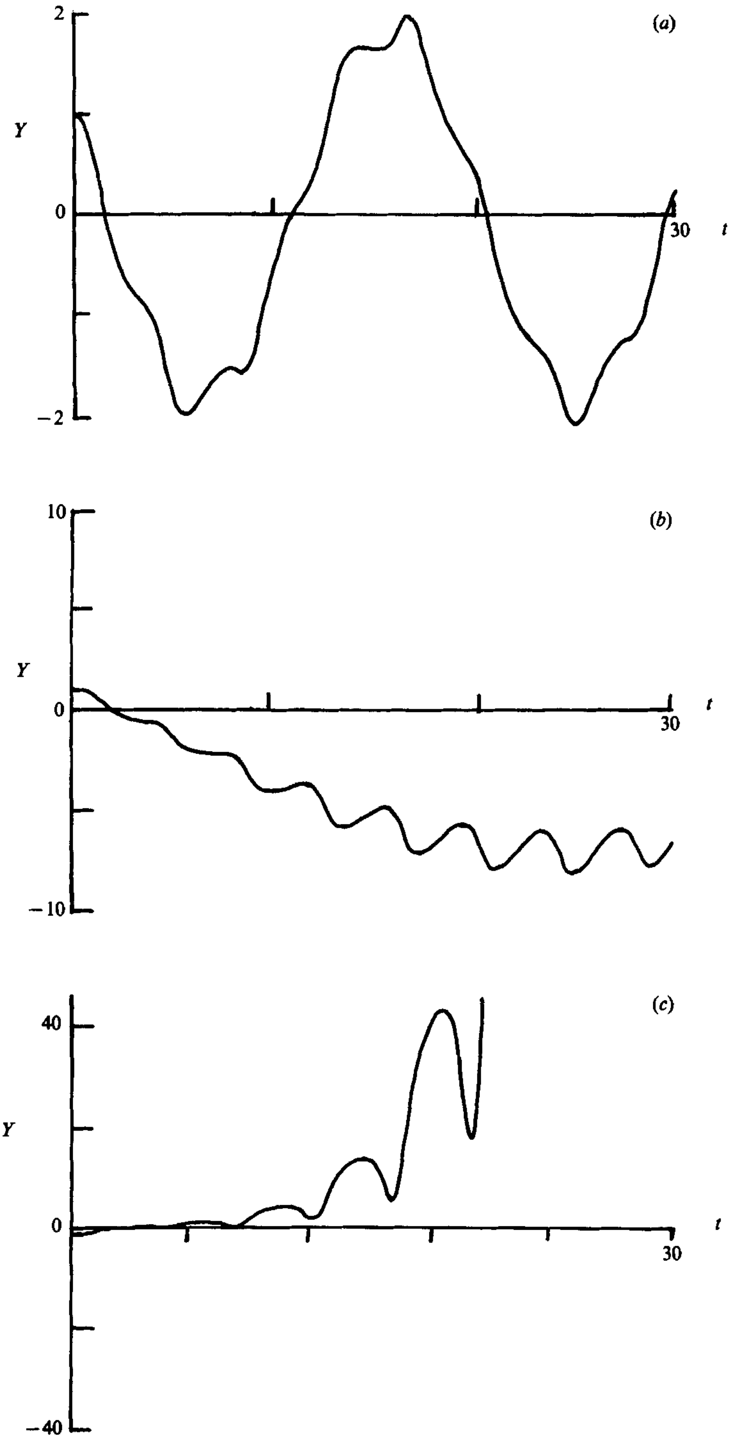

FIGURE 3. Typical results showing temporal evolution of inviscid disturbances. Y denotes disturbance amplitude and t is dimensionless time. Results shown are for $\Omega_1 = 0$ and $\epsilon = 0.6$ at three different wavenumbers. $\alpha_{10} = 0.8$, 0.72 and 0.6. (a) and (b) show doubly-periodic behaviour associated with imaginary Floquet exponents and (c) shows instability when the exponents are real.

with unbounded elliptical vortices; and the breakup of finite vortices may often have more to do with instabilities at the outer boundary than with the 'internal' instability mechanism discussed here.

Of relevance to the latter is work of Love (1893), who considered *irrotational* disturbances of finite elliptical vortex cores within otherwise irrotational flow: he found instability of such disturbances for eccentricities greater than $2\surd 2/3$. More recent attempts to determine the stability of elliptical flows are described by Cushman-Roisin (1986), Ripa (1985, 1987), Caprino & Salusti (1986) and Vladimirov & Tarasov (1985), but connections with the present work are unclear. For example, Ripa's eddies are of finite extent and subject to Coriolis force, his analysis is based on a shallow-water approximation and his unstable disturbances are assumed to have polynomial form in the horizontal coordinates.

In the light of our results, it is tempting to speculate that there may well be circumstances in which finite vortex structures are completely stabilized by sufficiently large rotation speeds in the opposite sense (because of the absolutely stable zone or because viscosity damps the narrower instability band at all realistic wavenumbers) whereas small negative rotation speeds may have the opposite effect (by allowing larger inviscid growth rates, as in figure 1). In geophysical situations, long-lasting elliptical eddies are certainly uncommon: a review of known cases (including Jupiter's red spot, of course) would be instructive.

The Floquet growth of plane waves described here is precisely the intensification of disturbance vorticity through stretching and tilting by the basic elliptical flow. In particular, it does not signal the breakup of an unbounded vortex (though this would certainly appear to be so in a numerical simulation); rather, it is a superposition of another growing flow upon it, since the disturbance, however strong it may become, has no influence on the basic flow. Also, although the unstable monochromatic waves are solutions of the full nonlinear equations, it should not be concluded that large-amplitude wavy flows represent the natural 'end state' of unstable elliptical eddies. As Haynes (1987) has shown in the simpler context of unidirectional shear flow on a beta plane, such plane-periodic disturbances may themselves be unstable (since the Rayleigh criterion for barotropic instability is satisfied locally). His numerical studies reveal that the planar disturbances do indeed break down into small-scale eddy structures of size equal to the wavelength of the planar flow. This of course suggests an effective mechanism for the creation of small-scale eddies (and so increased dissipation) in turbulent flows. Indeed, if the newly formed small eddies were themselves elliptical, the process would repeat itself down to still smaller scales. But there are many routes by which small-scale motions may be generated and it would be rash to fix on just this one.

4. Other basic states

We now return to the initial formulation of §2 to outline how similar solutions may be obtained for other basic states. These may be time-dependent and may be subject to other body forces.

We first note from (2.6) that when body forces are absent, time-dependent basic states satisfy

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}+\boldsymbol{S}^2 = \boldsymbol{M}(t) \quad \text{(symmetric)}; \quad \operatorname{tr}\boldsymbol{S} = 0. \tag{4.1}$$

If axes are chosen along principal rates of strain and if these strain rates are constant, the matrix $\boldsymbol{S}$ then takes the general form

$$\boldsymbol{S} = \begin{pmatrix} a & -\omega_3 & \omega_2 \\ \omega_3 & b & -\omega_1 \\ -\omega_2 & \omega_1 & -a-b \end{pmatrix}, \tag{4.2}$$

where the ω_i denoting the three components of basic vorticity satisfy

$$\omega_1(t) = \omega_1(0)\mathrm{e}^{at}, \quad \omega_2(t) = \omega_2(0)\mathrm{e}^{bt}, \quad \omega_3(t) = \omega_3(0)\mathrm{e}^{-(a+b)t} \tag{4.3a}$$

Correspondingly, variable strain rates $a(t)$, $b(t)$ yield

$$\omega_1(t) = \omega_1(0)\exp\int_0^t a(t')\,\mathrm{d}t' \quad \text{etc.} \tag{4.3b}$$

Clearly, the temporal evolution of the $\omega_i(t)$ is due to vortex stretching and contraction by the spatially uniform strain rates. The wavenumber $\boldsymbol{\alpha}(t)$ of planar disturbances is still given by (2.20) but it is no longer periodic. Solution for $\boldsymbol{\alpha}(t)$ with given $\boldsymbol{\alpha}(0)$ then allows solution of the disturbance equations (2.21) (with $\boldsymbol{\Omega}$ here equal to zero). Large-time solutions are likely to exhibit viscous decay because of ever-decreasing disturbance lengthscales; but periods of initial growth due to inviscid processes are sure to occur.

In general, the orientation of the principal axes of rate-of-strain need not remain fixed in an inertial reference frame. However, when these directions are variable, it is always possible to choose a rotating reference frame aligned along them. If the rotation rate of this frame is $-\boldsymbol{\Omega}(t)$, then corresponding centrifugal and Coriolis forces must be introduced into the formulation. It is then necessary to seek time-dependent solutions of an equation similar to (2.10) with imposed strain rates $a(t)$, $b(t)$ and rotation $\boldsymbol{\Omega}(t)$: note that an additional body-force term arising from $(\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t)\times\boldsymbol{x}$ will be present. This may be done by decomposing (2.10) into symmetric and antisymmetric parts, but details are not given here. Once the evolution of the basic flow has been determined, the linear disturbance equations analogous to (2.20) and (2.21) may be solved sequentially as before.

It seems likely that this method of solution will prove useful to examine the local evolution and small-scale instability of strained eddy structures in turbulence: cf. the small-scale instabilities found in the numerical simulations of Pierrehumbert (1986) and Metcalfe *et al.* (1987). In particular, there is considerable scope for merging the present analytical techniques with direct numerical simulations of such flows, by appropriate coordinate transformations. B. J. Bayly (private communication, 1988) has recently considered how his Floquet stability analysis may yield a local approximation for disturbance growth in more general flows.

We conclude with a brief account of other body forces for which similar exact solutions may be found.

4.1. *The beta-plane approximation*

In geophysical contexts, the variation with latitude of the Coriolis force may be important. The simplest approximation that incorporates this variation is the beta-plane model in which the rotation velocity $\boldsymbol{\Omega}$ is taken as a linear function of position,

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_0 + \mathbb{R}\boldsymbol{x}, \quad \mathbb{R} \equiv \{r_{ij}\}.$$

With basic velocity fields in the form (2.3), the resultant Coriolis force contains terms quadratic in the space coordinates. These may be incorporated into the pressure $p^{(0)}$ by addition of cubic terms in the x_i only if they are expressible in gradient form. This is so only for a restricted class of flows and is most clearly exemplified by considering the case $\boldsymbol{\Omega} = (0, 0, \Omega_3^{(0)} + \beta x_1)$. It may be verified that the permissible class of linearly varying basic flow then are those with matrix $\boldsymbol{S}$ of the form

$$\boldsymbol{S} = \begin{pmatrix} a & 0 & 0 \\ g & -2a & 0 \\ h & f & a \end{pmatrix}. \tag{4.4}$$

But equation (2.10) must also be satisfied: this requires that *steady* flows of the form (4.4) must have $h = f = 0$ and $g = -\Omega_3^{(0)}$. Such flows are combinations of an axisymmetric stagnation-point flow with axis of symmetry in the y-direction and an x-dependent plane Couette flow in the y-direction. The case of plane Couette flow alone has previously been studied by Boyd (1983).

Similar restrictions on the admissible basic flows were found by Craik (1988) when considering variable magnetohydrodynamic forces. Plane disturbances of such flows still satisfy (2.20), but the additional variable component of $\boldsymbol{\Omega}$ or magnetic field imposes a new restriction on the form of permissible disturbances: details need not be given here as they are similar to the magnetohydrodynamic case just mentioned.

4.2. *Buoyancy forces with density stratification*

In the simplest Boussinesq form, buoyancy due to density variations provides the body force per unit mass

$$\mathbb{F} = -g\hat{\boldsymbol{k}}\,(\rho/\rho_1), \tag{4.5}$$

where $\hat{\boldsymbol{k}}$ denotes the vertical unit vector and g denotes gravitational acceleration. Here, ρ is the variable density and ρ_1 a fixed reference density. The density may vary according to

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \kappa \nabla^2 \rho$$

where κ is a diffusion coefficient due to molecular or thermal effects. More generally, the density may be attributable to several different causes $c^{(i)}$: temperature, and various dissolved salts each with their own characteristic diffusion rates $\kappa^{(i)}$. In such cases, one may take

$$\rho = \sum_{j=1}^{N} a^{(j)} c^{(j)}, \quad \frac{\mathrm{D}c^{(j)}}{\mathrm{D}t} = \kappa^{(j)} \nabla^2 c^{(j)} \quad (j = 1, 2, \ldots N), \tag{4.6}$$

where the $a^{(j)}$ are known constants and the $c^{(j)}$ denote temperature or concentration of salts. Again, an exact description of several classes of basic state and disturbance may be given for such configurations. When basic states described by the velocity field $\mathcal{U}$ and density ρ or concentrations $c^{(i)}$ vary linearly in the space coordinates x_j, the Laplacians associated with diffusive processes are identically zero. Thus, basic states satisfy

$$\rho_1 \frac{\mathrm{D}\mathcal{U}}{\mathrm{D}t} = -\nabla p - \rho g \hat{\boldsymbol{k}}, \quad \frac{\mathrm{D}c^{(j)}}{\mathrm{D}t} = 0, \tag{4.7}$$

where now $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \mathcal{U} \cdot \nabla$.

On writing the basic states as

$$\mathscr{U}_i = \sigma_{ij} x_j + \mathscr{U}_i^{(0)}(t), \quad c^{(j)} = C_k^{(j)} x_k + \mathscr{C}_0^{(j)} \quad (\mathscr{C}_0^{(j)} \text{ constants}),$$

(cf. (2.3)), one finds that

$$\rho_1 \left(\frac{\mathrm{d}\mathscr{U}_i^{(0)}}{\mathrm{d}t} + \mathscr{U}_k^{(0)} \sigma_{ik} \right) = -\pi_i^{(0)} - g\delta_{3i} \sum_1^N a^{(j)} \mathscr{C}_0^{(j)}, \tag{4.8a}$$

$$\rho_1 \left(\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} + \boldsymbol{S}^2 \right) - \mathbb{C} = -\boldsymbol{M}(t) \quad \text{(symmetric)}, \tag{4.8b}$$

$$\frac{\mathrm{d}\boldsymbol{C}^{(j)}}{\mathrm{d}t} + \boldsymbol{S}^{\mathrm{T}} \boldsymbol{C}^{(j)} = 0, \quad \boldsymbol{C}^{(j)} = C_k^{(j)} \quad (k = 1, 2, 3), \tag{4.8c}$$

where the 3×3 matric $\mathbb{C}$ is

$$\mathbb{C} \equiv -g \sum_{j=1}^N a^{(j)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1^{(j)} & C_2^{(j)} & C_3^{(j)} \end{pmatrix}.$$

It is clear that each 'concentration vector' $\boldsymbol{C}^{(j)}$ evolves in just the same way as does the wavenumber $\boldsymbol{\alpha}(t)$ in (2.20). This is a consequence of the kinematic property that material planes remain plane throughout.

Admissible basic states satisfying (4.8) may be found and the corresponding evolution equations for planar disturbances constructed as in the preceding sections: details need not be given here. It suffices to observe that solutions may be found as above for many flows subject to buoyancy. For example, plane propagating internal gravity waves strained by basic flow fields may be studied. So too may be plane-periodic motions arising from buoyancy-driven instability associated with thermal and double (or multiple) diffusion, and subject also to straining by an ambient velocity distribution. The simplest basic states are those with purely horizontal velocity fields and uniform vertical stratification of concentrations (i.e. $C_1^{(j)} = C_2^{(j)} = 0$). It seems certain that plane internal gravity waves will exhibit a Floquet instability when strained by elliptical eddies, in the same manner as do the inertial waves studied in §3 above and the magnetohydrodynamic waves discussed by Craik (1988). However, detailed investigation of such situations is postponed until later.

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REFERENCES

Babiano, A., Basdevant, C., Legras, B. & Sadourny, R. 1987 Vorticity and passive-scalar dynamics in two-dimensional turbulence. *J. Fluid Mech.* **183**, 379–397.

Bayly, B. J. 1986 Three-dimensional instability of elliptical flow. *Phys. Rev. Lett.* **57**, 2160–2171.

Boyd, J. P. 1983 The continuous spectrum of linear Couette flow with the beta effect. *J. Atmos. Sci.* **40**, 2304–2308.

Caprino, S. & Salusti, E. 1986 Stability of oceanic eddies as vortex patches. Unpublished manuscript (*Ocean Modelling 72*).

Craik, A. D. D. 1988 A class of exact solutions in viscous incompressible magnetohydrodynamics. *Proc. R. Soc. Lond.* A **417**, 235–244.

Craik, A. D. D. & Criminale, W. O. 1986 Evolution of wavelike disturbances in shear flows: a class of exact solutions of the Navier–Stokes equations. *Proc. R. Soc. Lond.* A **406**, 13–26.

Criminale, W. O. 1985 Instabilities in a rotating fluid with constant shear. *J. Geophys. Res.* **90**, 862–868.

Criminale, W. O. & Cordova, J. Q. 1986 Effects of diffusion in the asymptotics of perturbations in stratified shear flow. *Phys. Fluids* **29**, 2054–2060.

Criminale, W. O. & Pinet-Plasencia, R. 1985 Unpublished manuscript 'Wave interaction with mean shear.' [See also Univ. of Washington M.S. thesis.]

Cushman-Roisin, B. 1986 Linear stability of large, elliptical warm-core rings. *J. Phys. Oceanogr.* **16**, 1158–1164.

Farrell, B. F. 1982 The initial growth of disturbances in a baroclinic flow. *J. Atmos. Sci.* **39**, 1663–1686.

Hartman, R. J. 1975 Wave propagation in s stratified shear flow. *J. Fluid Mech.* **71**, 89–104.

Haynes, P. H. 1987 On the instability of sheared disturbances. *J. Fluid Mech.* **175**, 463–478.

Kelvin, Lord 1887 Stability of fluid motion: rectilinear motion of viscous fluid between two parallel plates. *Phil. Mag.* **24** (5), 188–196.

Knobloch, E. 1984 On the stability of stratified plane Couette flow. *Geophys. Astrophys, Fluid Dynamics* **29**, 105–116.

Knobloch, E. 1985 The stability of non-separable barotropic and baroclinic shear flows. *Astrophy. and Space Sci.* **116**, 149–163.

Lagnado, R. R., Phan-Thien, N. & Leal, L. G. 1984 The stability of two-dimensional linear flows. *Phys. Fluids* **27**, 1094–1101.

Landman, M. J. & Saffman, P. G. 1987 The three-dimensional instability of strained vortices in a viscous fluid. *Phys. Fluids* **30**, 2339–2342.

Love, A. E. H. 1893 On the stability of certain vortex motions. *Proc. Lond. Math. Soc.* **25**, 18–42.

McWilliams, J. C. 1984 The emergence of isolated coherent vortices in turbulent flow. *J. Fluid Mech.* **146**, 21–43.

Metcalfe, R. W., Orszag, S. A., Brachet, M. E., Menon, S. & Riley, J. J. 1987 Secondary instability of a temporary growing mixed layer. *J. Fluid. Mech.* **184**, 207–243.

Phillips, O. M. 1966 *The Dynamics of the Upper Ocean.* Cambridge University Press.

Pierrehumbert, R. T. 1986 Universal short-wave instability of two-dimensional eddies in an inviscid fluid. *Phys. Rev. Lett.* **57**, 2157–2159.

Ripa, P. 1985 On the stability of Cushman–Roisin's warm eddy solution. Unpublished manuscript (*Ocean Modelling, 63*).

Ripa, P. 1987 On the stability of elliptical vortex solutions of the shallow-water equations. *J. Fluid Mech.* **183**, 343–363.

Shepherd, T. 1985 On the time development of small disturbances to plane Couette flow. *J. Atmos. Sci.* **42**, 1868–1871.

Tung, K. K. 1983 Initial-value problems for Rossby waves in a shear flow with critical level. *J. Fluid Mech.* **133**, 443–469.

Vladimirov, V. A. & Tarasov, V. F. 1985 Resonance instability of the flows with closed streamlines. In '*Laminar–Turbulent Transition: Proc. IUTAM Symp. Novosibirsk 1984*' (ed. V. V. Kozlov), pp. 717–722.

Yamagata, T. 1976 On the propagation of Rossby waves in a weak shear flow. *J. Met. Soc. Japan* **54**, 126–128.